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On Melnikov functions of a homoclinic loop through a nilpotent saddle for planar near-Hamiltonian systems

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Abstract

The first-order Melnikov function of a homoclinic loop through a nilpotent saddle for general planar near-Hamiltonian systems is considered. The asymptotic expansion of this Melnikov function and formulas for its first coefficients are given. The number of limit cycles which appear near the homoclinic loop is discussed by using the asymptotic expansion of the first-order Melnikov function. An example is presented as an application of the main results.

MSC: 34C05

Keywords: Nilpotent saddle point; Homoclinic loop; Melnikov function; Limit cycle; Bifurcation; Hamiltonian system

1. Introduction

Consider a planar near-Hamiltonian system

$$\begin{aligned} \dot{x} &= \bar{H}_y + \varepsilon f(x, y, a), \\ \dot{y} &= -\bar{H}_x + \varepsilon g(x, y, a), \end{aligned} \qquad (F)_\varepsilon$$

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where f, g and $\bar{H}$ are C^∞ functions, ε is a small parameter and a is a parameter vector with $a \in \mathbb{R}^n$. Suppose the unperturbed system $(F)_{\varepsilon=0}$ has a family of period orbits $\{\Gamma_h\} = \{(x, y)\colon\ \bar{H} = h\}$. Then, associated to a given perturbation of the above system, there exists a so-called the first-order Melnikov function of the following form

$$M(h, a) = \oint_{\Gamma_h} f\,dy - g\,dx. \tag{1}$$

An essential step for the study the weak Hilbert's 16th problem [1] is to calculate the number of isolated zeros of $M(h, a)$. There has been many works on this topic, for example, [3,12–14,19] and references therein. When h is taken as a critical value h_0, the graph of $\bar{H}(x, y) = h_0$ contains a singular point. For example, it may be a singular cycle consisting of singular points and regular orbits connecting them. The study on the asymptotic expansion of Melnikov function $M(h, a)$ at critical values is an interesting problem which is closely related to the weak Hilbert's 16th problem and limit cycle bifurcation near the corresponding singular cycle for near-Hamiltonian system $(F)_\varepsilon$. To the best our knowledge, the pioneer to discuss the expression of this Melnikov function at a homoclinic critical value is Robert Roussarie. Roussarie in [15] studied $(F)_{\varepsilon=0}$ with the level set $L = \{(x, y)\colon\ \bar{H}(x, y) = 0\}$, which is a singular cycle called a homoclinic loop. The homoclinic loop L passes through a hyperbolic saddle. He obtained the following asymptotic expansion as h tends to $h_0 = 0$,

$$M(h, a) = c_0(a) + c_1(a)h \ln|h| + c_2(a)h + c_3(a)h^2 \ln|h| + c_4(a)h^2 + \cdots.$$

From the asymptotic expansion, he also obtained the conditions that at most k limit cycles can be generated by L when the quantity $|\varepsilon| + |a - a_0|$ is very small, where $k \geqslant 1$ and $a_0 \in \mathbb{R}^n$ satisfy

$$c_j(a_0) = 0, \quad j = 0, \ldots, k-1, \qquad c_k(a_0) \neq 0.$$

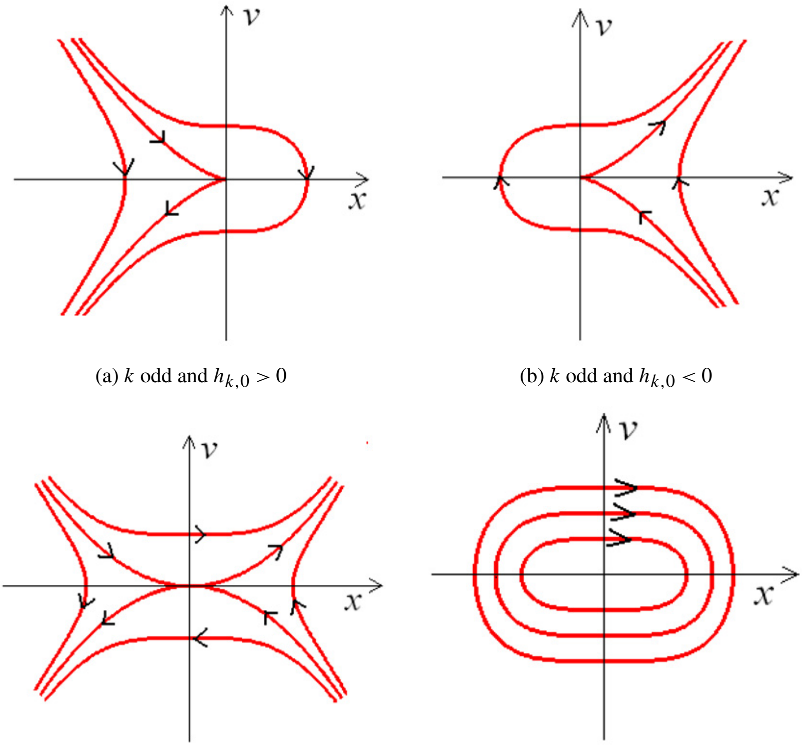
The formulas for computing c_0, c_1, c_2 were given in [9]. The formula for c_3 up to a constant was given in [11]. Recently, an exact formula for c_3 was found by [8]. A way to find three limit cycles near a homoclinic loop by using c_0, c_1, c_2 and c_3 was given in [11] and [8].

Thus, it is clear that the asymptotic expansion of $M(h, a)$ is an essential ingredient for the study of bifurcation problems of $(F)_\varepsilon$. However, except for the expansions near an elementary center or a homoclinic loop through a hyperbolic saddle point, it seems that few work has been done on the asymptotic expansions of $M(h, a)$ at such a critical value h_0 that the level curve $\bar{H}(x, y) = h_0$ contains a non-elementary singular point such as nilpotent singular points. To mention some recent works we will introduce some notation such as a cusp, a nilpotent saddle and a nilpotent center respectively below.

Suppose that the system $(F)_\varepsilon$ has a singular point at the origin, and the origin is a nilpotent singular point for $(F)_{\varepsilon=0}$, i.e. the Jacobian matrix of system $(F)_{\varepsilon=0}$ at the origin has a double zero eigenvalue and it is not zero matrix.

By the method in [10] and [18], system $(F)_\varepsilon$ can be transformed into the following system

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f_1(x, y, a), \\ \dot{y} &= -H_x + \varepsilon g_1(x, y, a), \end{aligned} \tag{$(E)_\varepsilon$}$$



(a) k odd and $h_{k,0} > 0$

(b) k odd and $h_{k,0} < 0$

(c) k even and $h_{k,0} < 0$

(d) k even and $h_{k,0} > 0$

Fig. 1. Nilpotent singular point under (3).

and the Hamiltonian has the form

$$H(x, y) = H_0(x) + y^2 H_2(x, y) \tag{2}$$

near the origin, where $H_0(x)$ and $H_2(x, y)$ are C^∞ with $H_0(x) = \sum_{j \geqslant 3} h_{j,0} x^j$, and $H_2(0, 0) \neq 0$. Suppose that $k \geqslant 3$ is an integer such that

$$h_{k,0} \neq 0, \qquad h_{j,0} = 0 \quad \text{for } j < k. \tag{3}$$

By (2) and a theorem in [17], it is easy to see that the origin is

(i) a cusp of system $(E)_{\varepsilon=0}$, if k is odd;
(ii) a center of system $(E)_{\varepsilon=0}$, if k is even with $h_{k,0} > 0$;
(iii) a nilpotent saddle of system $(E)_{\varepsilon=0}$, if k is even with $h_{k,0} < 0$.

See Fig. 1.

In the first and third cases, the level curve $H(x, y) = 0$ may contain a loop homoclinic to the origin, which is called a cuspidal loop or nilpotent saddle loop respectively. In the second case, the origin is called a nilpotent center. Limit cycle bifurcations near a nilpotent center and

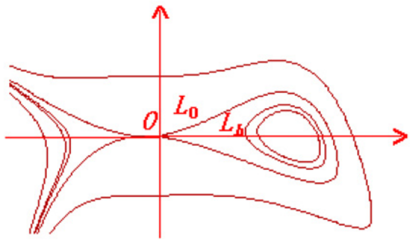


Fig. 2. The family of L_h.

a cuspidal loop for general plane system were studied in [7] and [10] respectively. For the perturbation of a cuspidal loop of non-Hamiltonian systems, see [5] and the references therein. In investigation of bifurcation problems with codimension more than 3, we often meet the problem of bifurcation of the nilpotent saddle loop, for example, bifurcations of a nilpotent saddle-node of codimension 4 (cf. [4,16]). However, to our knowledge there are not any works about the study on a nilpotent saddle loop through a nilpotent saddle for near-Hamiltonian system. In this paper, we will study the asymptotic expansion of the first-order Melnikov function at a critical value corresponding to a nilpotent saddle loop and give the computing formulas for the first coefficients appeared in the expansion. Then by this asymptotic expansion and these coefficients, we discuss the existence and number of limit cycles of $(E)_\varepsilon$.

This paper is organized as follows. In Section 2, we study the Melnikov function near the nilpotent saddle loop and give the asymptotic expansion of the first-order Melnikov function at the nilpotent saddle loop for $(E)_\varepsilon$. In Section 3. We discuss the existence and number of limit cycles of $(E)_\varepsilon$. We also provide an example to show the application of these theorems.

2. Asymptotic expansion of the first-order Melnikov function

As is shown in Section 1, we first consider the system $(E)_{\varepsilon=0}$ in this section, which satisfies the following conditions:

(C_1) The unperturbed system $(E)_{\varepsilon=0}$ has a nilpotent saddle point at $(0,0)$.

(C_2) The Hamiltonian of the system $(E)_{\varepsilon=0}$ has the following form

$$H(x,y)=\sum_{i\geqslant 2m} h_{i,0}x^i+y^2\sum_{i+j\geqslant 0} h_{i,j}x^iy^j\equiv H_0(x)+y^2H_2(x,y) \tag{4}$$

near the origin, where $h_{2m,0}<0,\ m\geqslant 2,\ H_2(0,0)=\frac{1}{2}$.
Without loss of generality, we take

$$h_{2m,0}=-\frac{1}{2m}. \tag{5}$$

(C_3) System $(E)_{\varepsilon=0}$ has a nilpotent saddle loop L_0 satisfying $H(x,y)=0$, and that for $0<-h\ll 1$, the equation $H(x,y)=h$ defines a smooth closed orbit L_h near L_0. See Fig. 2.

Definition 1. If the Hamiltonian of system $(E)_{\varepsilon=0}$ has the form (4) with $h_{2m,0} < 0, m \geqslant 2$, and $H_2(0,0) = \frac{1}{2}$, then we call the equilibrium of system $(E)_{\varepsilon=0}$ at $(0,0)$ a nilpotent saddle of order $m-1$.

Hereafter, we suppose the above conditions (C_1)–(C_3) are satisfied. We will prove

Theorem 1. *The first-order Melnikov function of system $(E)_\varepsilon$ near the nilpotent saddle loop L_0 has the following expansion*

$$M(h,a) = \sum_{j\geqslant 0} c_{j,m}^{(0)}(-h)^j + \sum_{k=1}^{2m-1} (-h)^{\frac{k+m}{2m}} \sum_{j\geqslant 0} c_{j,k,m}^{(1)}(-h)^j + \ln(-h)\sum_{j\geqslant 1} c_{j,m}^{(2)}(-h)^j,$$

where $0 < -h \ll 1$, $c_{j,m}^{(i)}$ $(i=0,2)$ and $c_{j,k,m}^{(1)}$ are functions depending on the parameter vector a.

We are going to split the proof of Theorem 1 into three steps in the following subsections. The first step is to use a formula for the derivative of the function $M(h,a)$ which is an integral of the divergence $f_{1x} + g_{1y}$ along L_h. And in this step the integral is to be divided into two parts $I_1(h)$ and $I_2(h)$, where $I_1(h)$ is taken over a local part of L_h near the origin. The part $I_2(h)$ is smooth. The main task is to study the expansion of $I_1(h)$ by using some transformation techniques, which is done in the second step. The last step is to find the expansion of $M(h,a)$ from the expansion of its derivative. The following is the detailed process.

2.1. Analysis of the function $\frac{\partial M(h,a)}{\partial h}$

Note that it is easier to obtain the expansion of the function of $\frac{\partial M(h,a)}{\partial h}$ than that of $M(h,a)$. In this subsection, we are firstly going to study the proprieties of the function $\frac{\partial M(h,a)}{\partial h}$ to find the expansion of it. In the following, let $M(h,a) = M(h)$ and $\frac{\partial M(h,a)}{\partial h} = M'(h)$.

From [6], we have

Lemma 2. *For the function $M'(h)$, it holds that*

$$M'(h) = \pm \oint_{L_h} (f_{1x} + g_{1y})\,dt,$$

where "$+$" (resp., "$-$") is taken when L_h expands (resp., shrinks) with h increasing.

In order to obtain the expansion of $M'(h)$ near the nilpotent saddle loop L_0, from Lemma 2, we only need to consider the smooth closed orbit L_h for $0 < -h \ll 1$, namely $L_h = \{(x,y) \mid H(x,y) = h,\ 0 < -h \ll 1\}$. Then we can write $L_h = L_{1h} \cup L_{2h}$, where $L_{1h} = \{(x,y) \mid H(x,y) = h,\ x_0(h) \leqslant x \leqslant x_a\} \subset U$, $x_0(h)$ satisfies $H_y(x_0(h),0) = 0$, $x_a > 0$ is a small constant, and U is a small rectangular neighborhood of the origin $O(0,0)$. See Fig. 3. Then for system $(E)_\varepsilon$, from Lemma 2, we have

$$M'(h) = \oint_{L_h} (f_{1x} + g_{1y})\,dt,$$

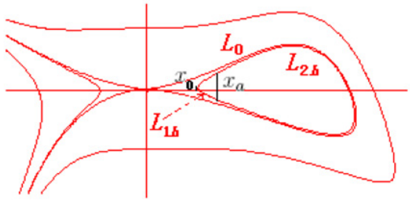


Fig. 3. L_{ih}.

which can be also rewritten as the following form

$$M'(h) = \int\limits_{L_{1h}} (f_{1x} + g_{1y})\,dt + \int\limits_{L_{2h}} (f_{1x} + g_{1y})\,dt \triangleq I_1(h) + I_2(h). \tag{6}$$

Obviously, we have

$$I_2(h) = I_2(0) + \Phi(h), \qquad \Phi(h) = O(h) \in C^\infty, \qquad I_2(0) = \int\limits_{L_{20}} (f_{1x} + g_{1y})\,dt, \tag{7}$$

where $L_{20} = \lim_{h\to 0} L_{2h}$.

2.2. Asymptotic expansion of the function $I_1(h)$

It follows from equality (7) that we can obtain the expansion of $M'(h)$ if we know the expansion of the function $I_1(h)$. Therefore, it is crucial to find the expansion of $I_1(h)$, which is also the right thing that we will investigate in the following.

Firstly, let us give some preliminary lemmas, which present some property of the function $I_1(h)$.

Let $d_{0,0} = (f_{1x} + g_{1y})|_{(0,0)}$. Then $f_{1x} + g_{1y} = d_{0,0} + O(|x, y|)$. Hence for system $(E)_\varepsilon$, from (6), we know

$$\begin{aligned} I_1(h) &= \int\limits_{L_{1h}} (f_{1x} + g_{1y})\,dt = \int\limits_{L_{1h}} \frac{d_{0,0} + O(|x, y|)}{H_y}\,dx = \int\limits_{L_{1h}} \frac{d_{0,0} + O(|x, y|)}{y(1 + O(|x, y|))}\,dx \\ &\equiv \int\limits_{L_{1h}} \frac{d_{0,0} + \sum_{i+j\geqslant 1} m_{i,j} x^i y^j}{y}\,dx. \end{aligned} \tag{8}$$

Clearly, there is no easy way for us to calculate integral (8) directly. In order to do that, let us introduce some important transformations, which is the key step and also provides us a way to complete the calculation of integral (8).

Consider the equation

$$H(x, y) = h \quad \text{or} \quad y^2 H_2(x, y) = w^2$$

near the origin, where $w=\sqrt{h-H_0(x)}$. By (4), the equation has solutions

$$w=\pm\frac{y}{\sqrt{2}}\sqrt{2H_2(x,y)}=\pm\frac{y}{\sqrt{2}}\bigl(1+O\bigl(|x,y|\bigr)\bigr). \tag{9}$$

Assume that $G_{\pm}(x,y,w)=\pm\frac{y}{\sqrt{2}}(1+O(|x,y|))-w$. Then the implicit function theorem implies that there exist two functions

$$y_+(x,w)=\sqrt{2}w\left(1+\sum_{i+j\geqslant 1}r^+_{i,j}x^iw^j\right),\qquad y_-(x,w)=-\sqrt{2}w\left(1+\sum_{i+j\geqslant 1}r^-_{i,j}x^iw^j\right) \tag{10}$$

such that

$$G_+\bigl(x,y^+,w\bigr)=0,\qquad G_-\bigl(x,y^-,w\bigr)=0. \tag{11}$$

Following the idea of [10,18], we can prove the following lemma.

Lemma 3. *The coefficients $r^{\pm}_{i,j}$ in* (10) *satisfy*

$$r^+_{i,j}=(-1)^jr^-_{i,j}. \tag{12}$$

Proof. Let

$$\tilde{w}=\frac{y}{\sqrt{2}}\bigl(1+O\bigl(|x,y|\bigr)\bigr). \tag{13}$$

Then (13) has a unique solution

$$y=\tilde{y}(x,\tilde{w})=\sqrt{2}\,\tilde{w}\left(1+\sum_{i+j\geqslant 1}r_{i,j}x^i\tilde{w}^j\right).$$

Hence, by (9) and (10)

$$y_+(x,w)=\tilde{y}(x,+w)=\sqrt{2}\,w\left(1+\sum_{i+j\geqslant 1}r_{i,j}x^iw^j\right), \tag{14}$$

$$y_-(x,w)=\tilde{y}(x,-w)=-\sqrt{2}\,w\left(1+\sum_{i+j\geqslant 1}(-1)^jr_{i,j}x^iw^j\right), \tag{15}$$

which implies that Eq. (12) holds. □

Note that we can write $L_{1h}=L^+_{1h}\cup L^-_{1h}$, where $L^{\pm}_{1h}=\{(x,y_{\pm}(x,w))\mid w=\sqrt{h-H_0(x)},\ x_0(h)\leqslant x\leqslant x_a\}$. Then, by (8), (12), (14) and (15), we have

$$
\begin{aligned}
I_1(h) &= \int_{x_0(h)}^{x_a} \frac{d_{0,0}+\sum_{i+j\geqslant 1} m_{i,j}x^i(y^+)^j}{y^+}\,dx + \int_{x_a}^{x_0(h)} \frac{d_{0,0}+\sum_{i+j\geqslant 1} m_{i,j}x^i(y^-)^j}{y^-}\,dx \\
&= \int_{x_0(h)}^{x_a} \frac{d_{0,0}+\sum_{i+j\geqslant 1} m_{i,j}x^i(\sqrt{2}\,w(1+\sum_{l+k\geqslant 1} r^+_{l,k}x^l w^k))^j}{\sqrt{2}\,w(1+\sum_{l+k\geqslant 1} r^+_{l,k}x^l w^k)}\,dx \\
&\quad + \int_{x_a}^{x_0(h)} \frac{d_{0,0}+\sum_{i+j\geqslant 1} m_{i,j}x^i(-\sqrt{2}\,w(1+\sum_{l+k\geqslant 1} r^-_{l,k}x^l w^k))^j}{-\sqrt{2}\,w(1+\sum_{l+k\geqslant 1} r^-_{l,k}x^l w^k)}\,dx \\
&\equiv \int_{x_0(h)}^{x_a} \frac{d_{0,0}+\sum_{i+j\geqslant 1} n^+_{i,j}x^i w^j}{\sqrt{2}\,w}\,dx + \int_{x_a}^{x_0(h)} \frac{d_{0,0}+\sum_{i+j\geqslant 1} n^-_{i,j}x^i w^j}{-\sqrt{2}\,w}\,dx. \qquad (16)
\end{aligned}
$$

Lemma 4. *The coefficients $n^{\pm}_{i,j}$ in* (16) *satisfy*

$$n^+_{i,j} = (-1)^j n^-_{i,j}. \qquad (17)$$

Proof. Let

$$\psi(x,w,r_{l,k}) = \frac{d_{0,0}+\sum_{i+j\geqslant 1} m_{i,j}x^i[\sqrt{2}\,w(1+\sum_{l+k\geqslant 1} r_{l,k}x^l w^k)]^j}{1+\sum_{l+k\geqslant 1} r_{l,k}x^l w^k}.$$

Then, we have

$$\psi\big(x,w,r^+_{l,k}\big) = d_{0,0} + \sum_{i+j\geqslant 1} n^+_{i,j}x^i w^j, \qquad \psi\big(x,-w,(-1)^k r^-_{l,k}\big) = d_{0,0} + \sum_{i+j\geqslant 1} n^-_{i,j}x^i w^j.$$

Recalling that $r^+_{l,k} = (-1)^k r^-_{l,k}$ from Lemma 3 gives

$$\psi\big(x,-w,(-1)^k r^-_{l,k}\big) = \psi\big(x,-w,r^+_{l,k}\big) = d_{0,0} + \sum_{i+j\geqslant 1} (-1)^j n^+_{i,j}x^i w^j.$$

It follows that $n^-_{i,j} = (-1)^j n^+_{i,j}$, which completes the proof. □

Recall that $H_0(x) = \sum_{i\geqslant 2m} h_{i,0}x^i = -\frac{x^{2m}}{2m}H_1(x)$, where $H_1(x) = 1 + O(x)$. Introducing a new variable u such that

$$u = x\big[H_1(x)\big]^{\frac{1}{2m}}, \qquad (18)$$

we know that $H_0(x) = -\frac{u^{2m}}{2m}$. And the inverse of (18) has the form

$$x = \sum_{j\geqslant 1} \phi_j u^j \equiv \phi(u), \qquad (19)$$

where ϕ_j is a function of $h_{i,j}$. Thus we have $H_0(\phi(u)) + \frac{u^{2m}}{2m} \equiv 0$. Then from (16) and (17), we have

$$\begin{aligned}
I_1(h) &= \int_{x_0(h)}^{x_a} \frac{d_{0,0} + \sum_{i+j\geqslant 1} n_{i,j}^{+} x^i w^j}{\sqrt{2}\, w}\, dx + \int_{x_0(h)}^{x_a} \frac{d_{0,0} + \sum_{i+j\geqslant 1} n_{i,j}^{-} x^i w^j}{\sqrt{2}\, w}\, dx \\
&= \int_{x_0(h)}^{x_a} \frac{\sqrt{2}(d_{0,0} + \sum_{i+2n\geqslant 1} n_{i,2n}^{+} x^i w^{2n})}{w}\, dx \\
&= \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} \frac{\sqrt{2}(d_{0,0} + \sum_{i+2n\geqslant 1} n_{i,2n}^{+} (\phi(u))^i w^{2n})}{w}\, d\big(\phi(u)\big) \\
&= \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} \frac{\sqrt{2}(d_{0,0} + \sum_{i+2n\geqslant 1} d_{i,2n} u^i w^{2n})}{w}\, du \\
&\equiv \sqrt{2}\bigg(d_{0,0}\, I_{0,0} + d_{1,0}\, I_{1,0} + \sum_{i+2n\geqslant 2} d_{i,2n} I_{i,2n}\bigg),
\end{aligned} \tag{20}$$

where

$$I_{i,2n} = \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} \frac{u^i w^{2n}}{w}\, du = \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} u^i \left(h + \frac{u^{2m}}{2m}\right)^{n-\frac{1}{2}} du, \quad w = \sqrt{h + \frac{u^{2m}}{2m}},$$

u_a is a constant and satisfies $\phi(u_a) = x_a$, and coefficient $d_{i,2n}$ is independent of (u, w).

Noting that integral (20) has infinite integral terms. It is impossible for us to calculate one by one. Then it is natural for one to investigate the properties of the integral to find a way to reduce the infinite terms into finite terms. Recall the following lemma.

Lemma 5.

$$\begin{aligned}
\int x^m \big(b_1 x^n + b_2\big)^p\, dx &= \frac{x^{m-n+1}}{b_1(np+m+1)} \big(b_1 x^n + b_2\big)^{p+1} \\
&\quad - \frac{(m-n+1) b_2}{b_1(np+m+1)} \int x^{m-n} \big(b_1 x^n + b_2\big)^p dx,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\int x^m \big(b_1 x^n + b_2\big)^p\, dx &= \frac{x^{m+1}}{np+m+1} \big(b_1 x^n + b_2\big)^{p} \\
&\quad + \frac{n\, p\, b_2}{np+m+1} \int x^m \big(b_1 x^n + b_2\big)^{p-1}\, dx.
\end{aligned} \tag{22}$$

One can easily find that it is enough for us to investigate only the expansion of the integral $I_{k,0}$ $(k = 0, 1, \dots, 2m-1)$, which can be used to express the other integrals appeared in (20). We state these by the following lemma.

Lemma 6. *For* $0 < -h \ll 1$, $I_{k,0}$ $(k = 0, \dots, 2m-1)$ *has the following expansion*

$$I_{k,0} = \Delta_{k,m}(-h)^{\frac{k-m+1}{2m}} + S_k \ln(-h) + \sum_{j \geqslant 0} p_{j,k,m}(-h)^j,$$

where $S_k = 0$ $(k \neq m-1)$, $S_{m-1} = -\frac{\sqrt{2m}}{2m}$, $\Delta_{k,m}$ *and* $p_{j,k,m}$ *are constants, and* $\Delta_{k,m} = 0$ *for* $k = m-1$ *and* $k = 2m-1$.

Proof. Let $h = -\frac{\lambda^{2m}}{2m}$ and $u = \frac{\lambda}{v}$. Then

$$\begin{aligned}
I_{k,0} &= \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} \frac{u^k}{\sqrt{h + \frac{u^{2m}}{2m}}}\,du = \int_{\lambda}^{u_a} \frac{\sqrt{2m}\,u^k}{\sqrt{u^{2m} - \lambda^{2m}}}\,du \\
&= \sqrt{2m}\lambda^{k-m+1} \int_{\frac{\lambda}{u_a}}^{1} \frac{1}{v^{k-m+2}\sqrt{1 - v^{2m}}}\,dv \\
&= \sqrt{2m}\lambda^{k-m+1} \left(\int_{\frac{1}{2}}^{1} \frac{1}{v^{k-m+2}\sqrt{1 - v^{2m}}}\,dv + \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} \frac{1}{v^{k-m+2}\sqrt{1 - v^{2m}}}\,dv \right).
\end{aligned}$$

Let

$$D^{(0)}_{k,m} = \int_{\frac{1}{2}}^{1} \frac{1}{v^{k-m+2}\sqrt{1 - v^{2m}}}\,dv. \tag{23}$$

Noting that

$$\frac{1}{\sqrt{1 - v^{2m}}} = \sum_{j \geqslant 0} \bar{A}_j\, v^{2mj}, \quad \text{where } \bar{A}_0 = 1,\ \bar{A}_j = \frac{(2j-1)!!}{2^j\, j!},\ j \geqslant 1,$$

we have

$$I_{k,0} = \sqrt{2m}\lambda^{k-m+1} \left(D^{(0)}_{k,m} + \sum_{j \geqslant 0} \bar{A}_j \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{2mj-k+m-2}\,dv \right). \tag{24}$$

The next step is to calculate the integral (24). In order to do so, we split it into three subcases according to its properties.

Case 1. $0 \leqslant k \leqslant m-2$.

For this case, we have $-k+m-2 \geqslant 0$. Then

$$I_{k,0} = \sqrt{2m}\lambda^{k-m+1}\left(D_{k,m}^{(0)} + \sum_{j\geqslant 0} \bar{A}_j \frac{v^{2mj-k+m-1}}{2mj-k+m-1}\bigg|_{\frac{\lambda}{u_a}}^{\frac{1}{2}}\right)$$
$$= \sqrt{2m}\lambda^{k-m+1}\left(D_{k,m}^{(0)} + \sum_{j\geqslant 0} \bar{A}_j \frac{(\frac{1}{2})^{2mj-k+m-1}}{2mj-k+m-1}\right)$$
$$- \sqrt{2m}\sum_{j\geqslant 0} \bar{A}_j \frac{(-h)^j(2m)^j}{(2mj-k+m-1)(u_a)^{2mj-k+m-1}}.$$

Noting that $2mj-m-k-1 \geqslant 0$. Then series $\sum_{j\geqslant 0} \bar{A}_j \frac{(\frac{1}{2})^{2mj-k+m-1}}{2mj-k+m-1}$ is convergent to a constant.

Let

$$D_{k,m}^{(1)} = D_{k,m}^{(0)} + \sum_{j\geqslant 0} \bar{A}_j \frac{(\frac{1}{2})^{2mj-k+m-1}}{2mj-k+m-1}, \quad 0 \leqslant k \leqslant m-2, \tag{25}$$

and

$$p_{j,k,m} = \frac{-\bar{A}_j(2m)^{j+\frac{1}{2}}}{(2mj-k+m-1)(u_a)^{2mj-k+m-1}}, \quad j \geqslant 0,\ 0 \leqslant k \leqslant m-2.$$

Hence

$$I_{k,0} = \sqrt{2m}D_{k,m}^{(1)}\lambda^{k-m+1} + \sum_{j\geqslant 0} p_{j,k,m}(-h)^j = \Delta_{k,m}(-h)^{\frac{k-m+1}{2m}} + \sum_{j\geqslant 0} p_{j,k,m}(-h)^j,$$

where

$$\Delta_{k,m} = D_{k,m}^{(1)}(2m)^{\frac{k+1}{2m}} \quad \text{for } 0 \leqslant k \leqslant m-2. \tag{26}$$

Case 2. $k = m-1$.

When $k = m-1$, we know $-k+m-2 = -1$. Then from (24) we can obtain

$$I_{k,0} = I_{m-1,0} = \sqrt{2m}\left(D_{m-1,m}^{(0)} + \sum_{j\geqslant 0} \bar{A}_j \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{2mj-1}\,dv\right)$$
$$= \sqrt{2m}\left(D_{m-1,m}^{(0)} + \bar{A}_0 \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{-1}dv + \sum_{j\geqslant 1} \bar{A}_j \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{2mj-1}\,dv\right)$$

$$
\begin{aligned}
&= \sqrt{2m}\left(D^{(0)}_{m-1,m} - \frac{1}{2m}\ln(2m) + \ln\frac{u_a}{2} + \sum_{j\geqslant 1} \bar{A}_j \frac{(\frac{1}{2})^{2mj}}{2mj} \right) \\
&\quad - \frac{\sqrt{2m}}{2m}\ln(-h) - \sqrt{2m}\sum_{j\geqslant 1} \bar{A}_j \frac{(-h)^j (2m)^j}{2mj\,(u_a)^{2mj}} \\
&\equiv S_{m-1}\ln(-h) + \sum_{j\geqslant 0} p_{j,k,m}(-h)^j,
\end{aligned}
$$

where

$$
S_{m-1} = -\frac{\sqrt{2m}}{2m}, \qquad p_{j,k,m} = -\frac{\sqrt{2m}\,\bar{A}_j (2m)^j}{2mj\,(u_a)^{2mj}}, \quad k = m-1,\ j \geqslant 1,
$$

$$
p_{0,k,m} = \sqrt{2m}\left(D^{(0)}_{m-1,m} - \frac{1}{2m}\ln(2m) + \ln\frac{u_a}{2} + \sum_{j\geqslant 1} \bar{A}_j \frac{(\frac{1}{2})^{2mj}}{2mj} \right), \quad k = m-1.
$$

Case 3. $m \leqslant k \leqslant 2m-1$.

For $m \leqslant k \leqslant 2m-1$, we know $-k+m-2 \leqslant -2$ and

$$
2mj - k + m - 2 \geqslant 2m - k + m - 2 \geqslant m - 1 \geqslant 0
$$

as $j \geqslant 1$. Then from (24)

$$
\begin{aligned}
I_{k,0} &= \sqrt{2m}\lambda^{k-m+1}\left(D^{(0)}_{k,m} + \bar{A}_0 \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{-k+m-2}\,dv + \sum_{j\geqslant 1}\bar{A}_j \int_{\frac{\lambda}{u_a}}^{\frac{1}{2}} v^{2mj-k+m-2}\,dv \right) \\
&= \sqrt{2m}\lambda^{k-m+1}\left(D^{(0)}_{k,m} + \frac{v^{m-k-1}}{m-k-1}\bigg|_{\frac{\lambda}{u_a}}^{\frac{1}{2}} + \sum_{j\geqslant 1}\bar{A}_j \frac{v^{2mj-k+m-1}}{2mj-k+m-1}\bigg|_{\frac{\lambda}{u_a}}^{\frac{1}{2}} \right) \\
&= \sqrt{2m}\lambda^{k-m+1}\left(D^{(0)}_{k,m} + \frac{\frac{1}{2}^{m-k-1}}{m-k-1} + \sum_{j\geqslant 1}\frac{\bar{A}_j \frac{1}{2}^{2mj-k+m-1}}{2mj-k+m-1} \right) + \sum_{j\geqslant 0} p_{j,k,m}(-h)^j \\
&\equiv \Delta_{k,m}(-h)^{\frac{k-m+1}{2m}} + \sum_{j\geqslant 0} p_{j,k,m}(-h)^j,
\end{aligned}
$$

where

$$
\Delta_{k,m} = (2m)^{\frac{k+1}{2m}}\left(D^{(0)}_{k,m} + \sum_{j\geqslant 0}\frac{\bar{A}_j \frac{1}{2}^{2mj-k+m-1}}{2mj-k+m-1} \right), \quad m \leqslant k \leqslant 2m-1, \tag{27}
$$

and

$$p_{j,k,m} = \frac{\sqrt{2m}}{u_a^{m-k-1}(k+1-m)}, \quad j=0,\ m \leqslant k \leqslant 2m-1,$$
$$p_{j,k,m} = \frac{-(2m)^{j+\frac{1}{2}}\bar{A}_j}{u_a^{2mj-k+m-1}(2mj-k+m-1)}, \quad j \geqslant 1,\ m \leqslant k \leqslant 2m-1.$$

Especially, for $k = 2m-1$, we have

$$\begin{aligned}
\Delta_{2m-1,m} &= D^{(0)}_{2m-1,m} - \frac{\frac{1}{2}^{-m}}{m} + \sum_{j \geqslant 1} \frac{\bar{A}_j \frac{1}{2}^{2mj-m}}{2mj-m} \\
&= \frac{2^m\sqrt{1-4^{-m}}}{m} - \frac{\frac{1}{2}^{-m}}{m} + \frac{1}{2^m(1+\sqrt{1-4^{-m}})} \\
&\equiv 0.
\end{aligned} \tag{28}$$

From the above analysis, we know Lemma 6 holds. □

Now we are ready to obtain the expansion of $I_1(h)$. Note that

$$I_{k,2n} = \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} \frac{u^k w^{2n}}{w}\,du = \int_{(-2mh)^{\frac{1}{2m}}}^{u_a} u^k \left(h + \frac{u^{2m}}{2m}\right)^{n-\frac{1}{2}} du.$$

By (21), for all $k \geqslant 2m$, we have

$$\begin{aligned}
I_{k,2n} &= -\frac{2m(-2m+k+1)\,h}{2mn+k-m+1} I_{k-2m,2n} + \frac{2mu_a^{k-2m+1}}{2mn+k-m+1}\left(h + \frac{u_a^{2m}}{2m}\right)^{n+\frac{1}{2}} \\
&\equiv -\frac{2m(-2m+k+1)\,h}{2mn+k-m+1} I_{k-2m,2n} + \sum_{j \geqslant 0} t_{j,k,m,n}(-h)^j,
\end{aligned} \tag{29}$$

where $t_{j,k,m,n}$ is a constant. Hence we can obtain the expansion of $I_{k,2n}$ ($k \geqslant 2m$) if that of $I_{0,2n}, \ldots, I_{2m-1,2n}$ are known. Further, by (22) we have

$$\begin{aligned}
I_{k,2n} &= \frac{(2mn-m)\,h}{2mn+k-m+1} I_{k,2n-2} + \frac{u_a^{k+1}}{2mn+k-m+1}\left(h + \frac{u_a^{2m}}{2m}\right)^{n-\frac{1}{2}} \\
&\equiv \frac{(2mn-m)\,h}{2mn+k-m+1} I_{k,2n-2} + \sum_{j \geqslant 0} s_{j,k,m,n}(-h)^j,
\end{aligned} \tag{30}$$

where $s_{j,k,m,n}$ is a constant. Therefore we can get the expansion of $I_{k,2n}$ if we know the expansion of $I_{k,0}$ with $k = 0, 1, \ldots, 2m-1$. Using the same method again, we can obtain the expansion of $I_{k,j}$ for each k and each j. Especially, taking $n=0$ and $n=1$ respectively, we have from (29) and (30)

$$I_{k,0}=\frac{2m(2m-k-1)h}{k-m+1}I_{k-2m,0}+\sum_{j\geqslant 0}t_{j,k,m,0}(-h)^j, \tag{31}$$

$$I_{k,2}=\frac{mh}{m+k+1}I_{k,0}+\sum_{j\geqslant 0}s_{j,k,m,1}(-h)^j. \tag{32}$$

Let

$$F_m=\sum_{k\geqslant 4m}d_{k,0}I_{k,0}+\sum_{k\geqslant 2m}d_{k,2}I_{k,2}+\sum_{k\geqslant 0,\,n\geqslant 2}d_{k,2n}I_{k,2n}.$$

Hence from (29)–(32) and Lemma 6 we have

$$\begin{aligned}
F_m&=h^2\Bigg(\sum_{k\geqslant 4m}\frac{4m^2(2m-k-1)(4m-k-1)d_{k,0}}{(k-m+1)(k-3m+1)}I_{k-4m,0}\\
&\quad+\sum_{k\geqslant 2m}\frac{2m^2(2m-k-1)d_{k,2}}{(m+k+1)(k-m+1)}I_{k-2m,0}+\sum_{j\geqslant 0}\alpha^{(0)}_{j,k,m}(-h)^j\\
&\quad+\sum_{k\geqslant 0,\,n\geqslant 2}\frac{(2mn-m)(2mn-3m)d_{k,2n}}{(2mn+k-m+1)(2mn+k-3m+1)}I_{k,2n-4}\Bigg)\\
&=\ln(-h)\sum_{j\geqslant 2}\alpha^{(1)}_{j,m}(-h)^j+\sum_{k=1-m}^{m-1}(-h)^{\frac{k}{2m}}\sum_{j\geqslant 2}\alpha^{(2)}_{j,k,m}(-h)^j,
\end{aligned}$$

where $\alpha^{(i)}_{j,k,m}$ $(i=0,1,2)$ is a constant depending on the parameter vector a.

Moreover, from Eqs. (20), (29)–(32) we have

$$\begin{aligned}
I_1(h)&=\sum_{k=0}^{2m-1}d_{k,0}I_{k,0}+\sum_{k=0}^{2m-1}(d_{k,2}I_{k,2}+d_{2m+k,0}I_{2m+k,0})+F_m\\
&=\sum_{k=0}^{2m-1}d_{k,0}I_{k,0}+h\sum_{k=0}^{2m-1}\left(\frac{m}{m+k+1}d_{k,2}-\frac{2m\,(k+1)}{m+k+1}d_{2m+k,0}\right)I_{k,0}\\
&\quad+\sum_{j\geqslant 0}\alpha^{(3)}_{j,k,m}(-h)^j+F_m\\
&\equiv\sum_{k=0}^{2m-1}d_{k,0}I_{k,0}+h\sum_{k=0}^{2m-1}U_{k,m}I_{k,0}+\sum_{j\geqslant 0}\alpha^{(3)}_{j,k,m}(-h)^j+F_m,
\end{aligned}$$

where

$$U_{k,m}=\frac{m}{m+k+1}d_{k,2}-\frac{2m\,(k+1)}{m+k+1}d_{2m+k,0}. \tag{33}$$

Noting that $\Delta_{2m-1,m}=0$. Lemma 6 and straightforward computing yield that

$$\begin{aligned}
I_1(h) &= \sum_{k=0}^{2m-2} d_{k,0}\Delta_{k,m}(-h)^{\frac{k-m+1}{2m}} - \sum_{k=0}^{2m-2} U_{k,m}\Delta_{k,m}(-h)^{\frac{k+m+1}{2m}} \\
&\quad + S_{m-1}(d_{m-1,0}+U_{m-1,m}h)\ln(-h) + \sum_{j\geqslant 0}\alpha^{(4)}_{j,m}(-h)^j + F_m \\
&= \sum_{k=1-m}^{m-1}(-h)^{\frac{k}{2m}}\sum_{j\geqslant 0}\alpha^{(5)}_{j,k,m}(-h)^j + \ln(-h)\sum_{j\geqslant 0}\alpha^{(6)}_{j,m}(-h)^j,
\end{aligned} \tag{34}$$

where $\alpha^{(4)}_{j,k,m}$, $\alpha^{(5)}_{j,k,m}$ and $\alpha^{(6)}_{j,k,m}$ are constants depending on the parameter vector a.

2.3. The expansion of $M(h)$

In this subsection, we are ready to give the expansion of $M(h)$ by using the expansion of $M'(h)$.

From Eqs. (6) and (20), we know

$$M'(h)=\sqrt{2}I_1(h)+I_2(0)+\Phi(h). \tag{35}$$

Therefore, for system $(E)_\varepsilon$, Lemmas 2 and 6 and Eqs. (34) and (35) yield

$$\begin{aligned}
M(h) &= \sqrt{2}\left(-\sum_{k=0}^{2m-2}\frac{2md_{k,0}\Delta_{k,m}}{k+m+1}(-h)^{\frac{k+m+1}{2m}} + \sum_{k=0}^{2m-2}\frac{2m\Delta_{k,m}U_{k,m}}{k+3m+1}(-h)^{\frac{k+3m+1}{2m}}\right) \\
&\quad - \sqrt{2}d_{m-1,0}S_{m-1}(-h)\ln(-h) + \frac{\sqrt{2}U_{m-1,m}S_{m-1}}{2}h^2\ln(-h) \\
&\quad + \ln(-h)\sum_{j\geqslant 3}\alpha^{(7)}_{j,m}(-h)^j + \sum_{k=1-m}^{m-1}(-h)^{\frac{k}{2m}}\sum_{j\geqslant 3}\alpha^{(8)}_{j,k,m}(-h)^j + \sum_{j\geqslant 0}\alpha^{(9)}_{j,m}(-h)^j,
\end{aligned}$$

which can be simplified into the following form

$$M(h)=\sum_{j\geqslant 0}c^{(0)}_{j,m}(-h)^j + \sum_{k=1}^{2m-1}(-h)^{\frac{k+m}{2m}}\sum_{j\geqslant 0}c^{(1)}_{j,k,m}(-h)^j + \ln(-h)\sum_{j\geqslant 1}c^{(2)}_{j,m}(-h)^j,$$

where $c^{(1)}_{j,k,m}$, $c^{(0)}_{j,m}$ and $c^{(2)}_{j,m}$ are constants depending on the parameter vector a. Thus the proof of Theorem 1 has been finished.

From the above analysis and Lemma 2, it is easy to obtain the following remarks.

Remark 7. The first $3m+1$ coefficients of the expansion of the first-order Melnikov function $M(h)$ in Theorem 1 satisfy the following

$$\begin{aligned}
c_{0,m}^{(0)} &= M(0) = \oint_{L_0} g_1\,dx - f_1\,dy,\\
c_{0,k,m}^{(1)} &= -\frac{2\sqrt{2}\,m d_{k-1,0}\Delta_{k-1,m}}{m+k}, \quad k=1,\dots,2m-1,\\
c_{1,k,m}^{(1)} &= \frac{2\sqrt{2}\,m U_{k-1,m}\Delta_{k-1,m}}{3m+k}, \quad k=1,\dots,m-1,\\
c_{1,m}^{(0)} &= \oint_{L_0} (f_{1x}+g_{1y})\,dt \quad \text{for } c_{0,1,m}^{(1)} = \cdots = c_{0,m-1,m}^{(1)} = 0,\\
c_{1,m}^{(2)} &= -\sqrt{2}d_{m-1,0}S_{m-1} = \frac{\sqrt{m}}{m}d_{m-1,0},\\
c_{2,m}^{(2)} &= \frac{\sqrt{2}\,U_{m-1,m}S_{m-1}}{2} = -\frac{\sqrt{m}}{4m}(d_{m-1,2}-2md_{3m-1,0}),
\end{aligned}$$

where expressions of $U_{k,m}$ and $\Delta_{k,m}$ can be found from Eqs. (23), (25)–(27) and (28). We also notice from Lemma 6 that $\Delta_{m-1,m}=0$, which implies $c_{0,m,m}^{(1)}=0$.

Remark 8. From Section 1, it is easy to see that the result of Theorem 1 still holds for system $(F)_\varepsilon$, whose unperturbed system is Hamiltonian system and it has a nilpotent saddle loop through the nilpotent saddle point $(0,0)$.

Remark 9. In this paper, we mainly investigate the degenerate case, namely the case for $m \geqslant 2$. In fact, the result of Theorem 1 also holds for case $m=1$, which corresponds to the case that the origin is a hyperbolic saddle point. That is to say, for system $(F)_\varepsilon$, if the unperturbed system is Hamiltonian system and the level set $H=0$ gives a homoclinic loop passing through a saddle point at $(0,0)$. Then using the method shown in the process of the proof of Theorem 1, we can also obtain the expansion of the first-order Melnikov function $M(h,a)$ has the following form near $h=0$

$$M(h,a) = c_0(a) + c_1(a)h\ln|h| + c_2(a)h + c_3(a)h^2\ln|h| + c_4(a)h^2 + \cdots,$$

which is the same as the result of Roussarie [15].

2.4. *Further study for the case $m=2$*

Let

$$c_1(a) = c_{0,m}^{(0)}, \qquad c_{k+1}(a) = c_{0,k,m}^{(1)}, \quad k=1,\dots,m-1,$$
$$c_{m+1}(a) = c_{1,m}^{(2)}, \qquad c_{m+2}(a) = c_{1,m}^{(0)},$$

$$c_{k+2}(a)=c^{(1)}_{0,k,m},\quad k=m+1,\ldots,2m-1,$$
$$c_{2m+1+k}(a)=c^{(1)}_{1,k,m},\quad k=1,\ldots,m-1,\qquad c_{3m+1}(a)=c^{(2)}_{2,m}.$$

Obviously, $c_1(a),\ldots,c_{3m+1}(a)$ are the first $3m+1$ coefficients of $M(h)$ in Theorem 1. Remark 7 presents the relationship between the coefficient $c_i(a)$ $(i=1,\ldots,3m+1)$ of the function $M(h)$ and system $(E)_\varepsilon$. And it is easy to see from the process of the proof of Theorem 1 that we can give the expressions of the first $3m+1$ coefficients of $M(h)$ for any positive integer m. Moreover, we can also compute the values $\Delta_{k,m}$ in Lemma 6 and $c_i(a)$ $(i=1,\ldots,3m+1)$ in Remark 7 by using mathematic tools such as Maple 10.0. In the following, we will take $m=2$ as an example to show how to calculate these values and give the relationship between the coefficient $c_i(a)$ and the parameter vector a for $m=2$. For simplicity, in the following, let $c_i(a)=c_i$.

For $m=2$, from (4) and (5), it follows that $H(x,y)$ has the following formal expansion at the origin

$$H(x,y)=-\frac{1}{4}x^4+\sum_{j\geqslant 5}h_{j,0}x^j+y^2\sum_{i+j\geqslant 0}h_{i,j}x^iy^j.$$

Then for $m=2$, we have the following lemma.

Lemma 10. *For $0<-h\ll 1$, $I_{k,0}$ $(k=0,1,2,3)$ has the following expression*

$$I_{k,0}=\Delta_{k,2}(-h)^{\frac{k-1}{4}}+S_k\ln(-h)+\sum_{j\geqslant 0}p_{j,k,2}(-h)^j, \tag{36}$$

and

$$S_k=0\quad(k=0,2,3),\qquad S_1=-\frac{1}{2},\qquad \Delta_{0,2}>0,\qquad \Delta_{1,2}=\Delta_{3,2}=0,\qquad \Delta_{2,2}<0. \tag{37}$$

Proof. By Lemma 6, it is obvious that expansion (36) holds and $S_k=0$ $(k=0,2,3)$, $S_1=-\frac{1}{2}$, $\Delta_{1,2}=\Delta_{3,2}=0$. In the following, we focus on the proof of the other values in (37). Let $h=-\frac{\lambda^4}{4}$ and $u=\frac{\lambda}{v}$. Then from Lemma 6 we know

$$I_{0,0}=\int\limits_{(-4h)^{\frac{1}{4}}}^{u_a}\frac{1}{\sqrt{h+\frac{u^4}{4}}}\,du=\Delta_{0,2}(-h)^{-\frac{1}{4}}+\sum_{j\geqslant 0}p_{j,0,2}(-h)^j,$$

where

$$\Delta_{0,2}=2D^{(1)}_{0,2}=D^{(0)}_{0,2}+\sqrt{2}\sum_{j\geqslant 1}\bar{A}_j\frac{(\frac{1}{2})^{4j+1}}{4j+1},$$

$$D^{(0)}_{0,2}=\int\limits_{\frac{1}{2}}^{1}\frac{1}{\sqrt{1-v^4}}\,dv\doteq 0.8078193345$$

and

$$p_{j,0,2} = -\sum_{j\geqslant 0} \frac{2^{2j+1}\bar{A}_j}{(4j+1)(u_a)^{4j+1}}.$$

And

$$\begin{aligned}\Delta_{0,2} &= \sqrt{2}D_{0,2}^{(0)} + \frac{\sqrt{2}}{2} + \sqrt{2}\sum_{j\geqslant 1}\bar{A}_j \frac{(\frac{1}{2})^{4j+1}}{4j+1}\\ &= \sqrt{2}D_{0,2}^{(0)} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{320}F\left(\left[1, \frac{5}{4}, \frac{3}{2}\right], \left[2, \frac{9}{4}\right], \frac{1}{16}\right) > 0,\end{aligned}$$

since

$$\frac{1}{320}F\left(\left[1, \frac{5}{4}, \frac{3}{2}\right], \left[2, \frac{9}{4}\right], \frac{1}{16}\right) \doteq 0.003209443178$$

is a hypergeometric series.

Further, from Lemma 6

$$I_{1,0} = \int_{(-4h)^{\frac{1}{4}}}^{u_a} \frac{u}{\sqrt{h + \frac{u^4}{4}}}\,du = -\frac{1}{2}\ln(-h) + \sum_{j\geqslant 0} p_{j,1,2}(-h)^j$$

and

$$I_{2,0} = \int_{(-4h)^{\frac{1}{4}}}^{u_a} \frac{u^2}{\sqrt{h + \frac{u^4}{4}}}\,du \equiv \Delta_{2,2}(-h)^{\frac{1}{4}} + \sum_{j\geqslant 0} p_{j,2,2}(-h)^j,$$

where

$$\begin{aligned}\Delta_{2,2} &= 2\sqrt{2}D_{2,2}^{(0)} - 4\sqrt{2} + 2\sqrt{2}\sum_{j\geqslant 1}\bar{A}_j \frac{(\frac{1}{2})^{4j-1}}{4j-1}\\ &= 2\sqrt{2}D_{2,2}^{(0)} - 4\sqrt{2} + \frac{2\sqrt{2}}{48}F\left(\left[\frac{3}{4}, 1, \frac{3}{2}\right], \left[\frac{7}{4}, 2\right], \frac{1}{16}\right) < 0,\end{aligned}$$

since $D_{2,2}^{(0)} = \int_{\frac{1}{2}}^{1} \frac{1}{v^2\sqrt{1-v^4}}\,dv \doteq 1.379663569$ and

$$\frac{1}{48}F\left(\left[\frac{3}{4}, 1, \frac{3}{2}\right], \left[\frac{7}{4}, 2\right], \frac{1}{16}\right) \doteq 0.021266314$$

is a hypergeometric series.

From the above analysis, we know Lemma 10 holds. □

Form Theorem 1, it is obvious that we have

Theorem 2. *Let* (4) *be satisfied and* $m = 2$. *Then for system* $(E)_\varepsilon$, *near the value* $h = 0$ $(0 < -h \ll 1)$ *corresponding to nilpotent saddle loop* L_0 *through a nilpotent saddle point* $(0, 0)$, *we have*

$$\begin{aligned}(1)\quad M(h) &= \sum_{j\geqslant 0} c^{(0)}_{j,2}(-h)^j + (-h)^{\frac{3}{4}} \sum_{j\geqslant 0} c^{(1)}_{j,1,2}(-h)^j + (-h)^{\frac{5}{4}} \sum_{j\geqslant 0} c^{(1)}_{j,3,2}(-h)^j \\ &\quad + \ln(-h) \sum_{j\geqslant 1} c^{(2)}_{j,2}(-h)^j \\ &= c_1 + c_2(-h)^{\frac{3}{4}} + c_3(-h)\ln(-h) + c_4(-h) + c_5(-h)^{\frac{5}{4}} + c_6(-h)^{\frac{7}{4}} \\ &\quad + c_7(-h)^2\ln(-h) + \sum_{j\geqslant 2} c^{(0)}_{j,2}(-h)^j + (-h)^{\frac{3}{4}} \sum_{j\geqslant 2} c^{(1)}_{j,1,2}(-h)^j \\ &\quad + (-h)^{\frac{5}{4}} \sum_{j\geqslant 2} c^{(1)}_{j,3,2}(-h)^j + \ln(-h) \sum_{j\geqslant 3} c^{(2)}_{j,2}(-h)^j;\end{aligned}$$

(2) *the coefficients* $c_1, \dots, c_7$ *in the above asymptotic expansion satisfy*

$$\begin{aligned}&c_1 = M(0), \qquad c_2 = -\frac{4\sqrt{2}d_{0,0}\Delta_{0,2}}{3}, \qquad c_3 = \frac{\sqrt{2}d_{1,0}}{2}, \\ &c_4 = \oint_{L_0} (f_{1x} + g_{1y})\,dt \quad \text{for } c_2 = c_3 = 0, \\ &c_5 = -\frac{4\sqrt{2}d_{2,0}\Delta_{2,2}}{5}, \\ &c_6 = \frac{8\sqrt{2}(d_{0,2} - 2d_{4,0})\Delta_{0,2}}{21}, \qquad c_7 = \frac{\sqrt{2}(2d_{5,0} - \frac{d_{1,2}}{2})}{4},\end{aligned}$$

where the expressions of $d_{i,0}$ $(i = 1, 2, 4, 5)$ *and* $d_{i,2}$ $(i = 1, 2)$ *are given in the following theorem.*

Theorem 3. *Let* $m = 2$ *and*

$$f_1(x, y, a) = \sum_{i+j\geqslant 1} a_{i,j}x^i y^j, \qquad g_1(x, y, a) = \sum_{i+j\geqslant 1} b_{i,j}x^i y^j.$$

Then by using Maple 10.0, *we can obtain the expressions of constants* $m_{i,j}$ *in* (8), $r^{\pm}_{i,j}$ *in* (12), $n^{\pm}_{i,j}$ *in* (17) *and* $d_{i,j}$ *in* (20). *Especially, we have*

$$\begin{aligned}&d_{1,0} = (2h_{5,0} - h_{1,2})(a_{1,0} + b_{0,1}) + b_{1,1} + 2a_{2,0}, \\ &d_{0,2} = \left(15h^2_{0,3} - 6h_{0,4}\right)(a_{1,0} + b_{0,1}) - 6h_{0,3}(a_{1,1} + 2b_{0,2}) + 2a_{1,2} + 6b_{0,3},\end{aligned}$$

$$
\begin{aligned}
d_{2,0} = {} & \left(-3h_{5,0}h_{1,2} + \frac{3}{2}h_{1,2}^2 + 3h_{6,0} + \frac{21}{2}h_{5,0}^2 - h_{2,2}\right)(a_{1,0} + b_{0,1}) + 3a_{3,0} + b_{2,1} \\
& + (3h_{5,0} - h_{1,2})(2a_{2,0} + b_{1,1}), \\
d_{1,2} = {} & 6(5h_{1,2}h_{0,3} - h_{1,3} - 2h_{5,0}h_{0,3})(a_{1,1} + 2b_{0,2}) + (4h_{5,0} - 6h_{1,2})(a_{1,2} + 3b_{0,3}) \\
& + \big(15h_{0,3}^2 - 6h_{0,4}\big)(2a_{2,0} + b_{1,1}) + \big(30h_{1,2}h_{0,4} - 105h_{1,2}h_{0,3}^2 - 6h_{1,4} \\
& + 30h_{5,0}h_{0,3}^2 + 30h_{1,3}h_{0,3} - 12h_{5,0}h_{0,4}\big)(a_{1,0} + b_{0,1}) + 6b_{1,3} \\
& + 4a_{2,2} - 12h_{0,3}(a_{2,1} + b_{1,2}), \\
d_{4,0} = {} & \left(5h_{8,0} - \frac{25}{2}h_{5,0}h_{1,2}^3 - 5h_{5,0}h_{3,2} + \frac{15}{2}h_{1,2}^2h_{6,0} - \frac{45}{2}h_{2,2}h_{5,0}^2 - \frac{195}{2}h_{1,2}h_{5,0}^3\right. \\
& + \frac{585}{2}h_{6,0}h_{5,0}^2 - 5h_{1,2}h_{7,0} + \frac{3315}{8}h_{5,0}^4 + \frac{45}{2}h_{6,0}^2 - h_{4,2} + \frac{3}{2}h_{2,2}^2 - 5h_{2,2}h_{6,0} \\
& + 3h_{3,2}h_{1,2} - 45h_{5,0}h_{1,2}h_{6,0} - \frac{15}{2}h_{2,2}h_{1,2}^2 + \frac{135}{4}h_{1,2}^2h_{5,0}^2 + 15h_{5,0}h_{2,2}h_{1,2} \\
& \left. + \frac{135}{4}h_{1,2}^2h_{5,0}^2 + \frac{35}{8}h_{1,2}^4 + 45h_{7,0}h_{5,0}\right)(a_{1,0} + b_{0,1}) + (5h_{5,0} - h_{1,2})(4a_{4,0} + b_{3,1}) \\
& + \left(5h_{6,0} + \frac{45}{2}h_{5,0}^2 - 5h_{5,0}h_{1,2} + \frac{3}{2}h_{1,2}^2 - h_{2,2}\right)(3a_{3,0} + b_{2,1}) \\
& + \left(5h_{7,0} - \frac{5}{2}h_{1,2}^3 - 5h_{5,0}h_{2,2} + 45h_{6,0}h_{5,0} + 3h_{2,2}h_{1,2} - h_{3,2} - \frac{45}{2}h_{1,2}h_{5,0}^2\right. \\
& \left. - 5h_{1,2}h_{6,0} + \frac{195}{2}h_{5,0}^3 + \frac{15}{2}h_{5,0}h_{1,2}^2\right)(2a_{2,0} + b_{1,1}) + 5a_{5,0} + b_{4,1}, \\
d_{5,0} = {} & \left(-h_{2,2} - 6h_{5,0}h_{1,2} + 30h_{5,0}^2 + 6h_{6,0} + \frac{3}{2}h_{1,2}^2\right)(4a_{4,0} + b_{3,1}) \\
& + \left(9h_{5,0}h_{1,2}^2 + 60h_{6,0}h_{5,0} - h_{3,2} + 3h_{2,2}h_{1,2} - 6h_{5,0}h_{2,2} - \frac{5}{2}h_{1,2}^3\right. \\
& \left. - 30h_{1,2}h_{5,0}^2 - 6h_{1,2}h_{6,0} + 140h_{5,0}^3 + 6h_{7,0}\right)(3a_{3,0} + b_{2,1}) \\
& + \left(18h_{5,0}h_{2,2}h_{1,2} - h_{4,2} - 6h_{5,0}h_{3,2} - \frac{15}{2}h_{2,2}h_{1,2}^2 - 6h_{1,2}h_{7,0} + 60h_{7,0}h_{5,0}\right. \\
& - 140h_{1,2}h_{5,0}^3 - 60h_{5,0}h_{1,2}h_{6,0} + 3h_{3,2}h_{1,2} + \frac{35}{8}h_{1,2}^4 - 15h_{5,0}h_{1,2}^3 \\
& - 30h_{2,2}h_{5,0}^2 + 45h_{1,2}^2h_{5,0}^2 + \frac{3}{2}h_{2,2}^2 + 9h_{1,2}^2h_{6,0} + 420h_{6,0}h_{5,0}^2 \\
& \left. - 6h_{2,2}h_{6,0} + 30h_{6,0}^2 + 6h_{8,0} + 630h_{5,0}^4\right)(2a_{2,0} + b_{1,1}) \\
& + \Big(90h_{2,2}h_{1,2}h_{5,0}^2 + 18h_{2,2}h_{1,2}h_{6,0} - 45h_{5,0}h_{2,2}h_{1,2}^2 - 60h_{2,2}h_{6,0}h_{5,0}
\end{aligned}
$$

$$
\begin{aligned}
&-420h_{1,2}h_{6,0}h_{5,0}^2+90h_{1,2}^2h_{6,0}h_{5,0}-60h_{1,2}h_{7,0}h_{5,0}+18h_{5,0}h_{1,2}h_{3,2}\\
&-h_{5,2}+6h_{9,0}+2772h_{5,0}^5+420h_{5,0}h_{6,0}^2+\frac{35}{2}h_{2,2}h_{1,2}^3-\frac{15}{2}h_{1,2}h_{2,2}^2\\
&-\frac{15}{2}h_{3,2}h_{1,2}^2+3h_{2,2}h_{3,2}+60h_{7,0}h_{6,0}+3h_{4,2}h_{1,2}+\frac{105}{4}h_{5,0}h_{1,2}^4\\
&-6h_{5,0}h_{4,2}+9h_{5,0}h_{2,2}^2-15h_{1,2}^3h_{6,0}-6h_{3,2}h_{6,0}-75h_{1,2}^3h_{5,0}^2-6h_{2,2}h_{7,0}\\
&+9h_{1,2}^2h_{7,0}-30h_{3,2}h_{5,0}^2+210h_{1,2}^2h_{5,0}^3-30h_{1,2}h_{6,0}^2-140h_{2,2}h_{5,0}^3\\
&-630h_{1,2}h_{5,0}^4-6h_{1,2}h_{8,0}+2520h_{6,0}h_{5,0}^3+60h_{8,0}h_{5,0}+420h_{7,0}h_{5,0}^2\\
&-\frac{63}{8}h_{1,2}^5\Big)(a_{1,0}+b_{0,1})+(6h_{5,0}-h_{1,2})(5a_{5,0}+b_{4,1})+6a_{6,0}+b_{5,1}.
\end{aligned}
$$

Proof. Let $f_0=\frac{\partial f_1}{\partial x}$, $g_0=\frac{\partial g_1}{\partial y}$, $\operatorname{div}=f_0+g_0$ and

$$
S(x,y)=d_{0,0}+\sum_{i+j\geqslant 1}m_{i,j}x^iy^j,\qquad P_1=\operatorname{div}-S(x,y)H_y(x,y)/y.
$$

Then from $P_1=0$, using Maple 10.0, we have

$$
\begin{aligned}
m_{1,0}&=-2h_{1,2}(a_{1,0}+b_{0,1})+b_{1,1}+2a_{2,0},\\
m_{0,1}&=-3h_{0,3}(a_{1,0}+b_{0,1})+a_{1,1}+2b_{0,2},\\
m_{2,0}&=\big(4h_{1,2}^2-2h_{2,2}\big)(a_{1,0}+b_{0,1})+b_{2,1}-2h_{1,2}b_{1,1}-4h_{1,2}a_{2,0}+3a_{3,0},\\
m_{0,2}&=\big(-4h_{0,4}+9h_{0,3}^2\big)(a_{1,0}+b_{0,1})+a_{1,2}-3h_{0,3}a_{1,1}-6h_{0,3}b_{0,2}+3b_{0,3},\\
m_{1,1}&=(-3h_{1,3}+12h_{1,2}h_{0,3})(a_{1,0}+b_{0,1})+2a_{2,1}+2b_{1,2}-4h_{1,2}b_{0,2}\\
&\quad-6h_{0,3}a_{2,0}-2h_{1,2}a_{1,1}-3h_{0,3}b_{1,1},\\
m_{3,0}&=\big(4h_{1,2}^2-2h_{2,2}\big)(2a_{2,0}+b_{1,1})+\big(8h_{2,2}h_{1,2}-2h_{3,2}-8h_{1,2}^3\big)(a_{1,0}+b_{0,1})\\
&\quad+b_{3,1}-2h_{1,2}b_{2,1}+4a_{4,0}-6h_{1,2}a_{3,0},\\
m_{0,3}&=\big(9h_{0,3}^2-4h_{0,4}\big)(a_{1,1}+2b_{0,2})+\big(24h_{0,3}h_{0,4}-5h_{0,5}-27h_{0,3}^3\big)(a_{1,0}+b_{0,1})\\
&\quad+a_{1,3}-3h_{0,3}a_{1,2}+4b_{0,4}-9h_{0,3}b_{0,3},\\
m_{1,2}&=(-3h_{1,3}+12h_{1,2}h_{0,3})(a_{1,1}+2b_{0,2})+\big(9h_{0,3}^2-4h_{0,4}\big)(2a_{2,0}+b_{1,1})\\
&\quad+\big(-54h_{1,2}h_{0,3}^2-4h_{1,4}+18h_{1,3}h_{0,3}+16h_{1,2}h_{0,4}\big)(a_{1,0}+b_{0,1})\\
&\quad+2a_{2,2}+3b_{1,3}-6h_{1,2}b_{0,3}-6h_{0,3}a_{2,1}-2h_{1,2}a_{1,2}-6h_{0,3}b_{1,2},\\
m_{2,1}&=\big(4h_{1,2}^2-2h_{2,2}\big)(a_{1,1}+2b_{0,2})+(-3h_{1,3}+12h_{1,2}h_{0,3})(2a_{2,0}+b_{1,1})\\
&\quad+\big(12h_{0,3}h_{2,2}+12h_{1,3}h_{1,2}-3h_{2,3}-36h_{0,3}h_{1,2}^2\big)(a_{1,0}+b_{0,1})\\
&\quad+3a_{3,1}+2b_{2,2}-9h_{0,3}a_{3,0}-4h_{1,2}a_{2,1}-4h_{1,2}b_{1,2}-3h_{0,3}b_{2,1},
\end{aligned}
$$

$$
\begin{aligned}
m_{4,0} &= \big(4h_{1,2}^2 - 2h_{2,2}\big)(3a_{3,0} + b_{2,1}) + \big(8h_{2,2}h_{1,2} - 8h_{1,2}^3 - 2h_{3,2}\big)(2a_{2,0} + b_{1,1}) \\
&\quad + \big(-24h_{2,2}h_{1,2}^2 + 8h_{3,2}h_{1,2} + 16h_{1,2}^4 - 2h_{4,2} + 4h_{2,2}^2\big)(a_{1,0} + b_{0,1}) \\
&\quad - 2h_{1,2}(4a_{4,0} + b_{3,1}) + 5a_{5,0} + b_{4,1}, \\
m_{5,0} &= \big(4h_{1,2}^2 - 2h_{2,2}\big)(4a_{4,0} + b_{3,1}) + \big(8h_{2,2}h_{1,2} - 8h_{1,2}^3 - 2h_{3,2}\big)(3a_{3,0} + b_{2,1}) \\
&\quad + \big(-24h_{2,2}h_{1,2}^2 + 8h_{3,2}h_{1,2} + 16h_{1,2}^4 - 2h_{4,2} + 4h_{2,2}^2\big)(2a_{2,0} + b_{1,1}) \\
&\quad + \big(8h_{4,2}h_{1,2} - 24h_{3,2}h_{1,2}^2 + 64h_{2,2}h_{1,2}^3 - 2h_{5,2} - 24h_{1,2}h_{2,2}^2 - 32h_{1,2}^5 \\
&\quad + 8h_{3,2}h_{2,2}\big)(a_{1,0} + b_{0,1}) + b_{5,1} + 6a_{6,0} - 10h_{1,2}a_{5,0} - 2h_{1,2}b_{4,1}.
\end{aligned}
$$

For $y_+(x,w) = \sqrt{2}\,w(1 + \sum_{i+j\geqslant 1} r_{i,j}^+ x^i w^j)$, let $P_2 = w^2 - y_+^2 H_2(x, y_+)$. Then from $P_2 = 0$, we can obtain

$$
\begin{aligned}
& r_{0,1}^+ = -h_{0,3}\sqrt{2}, \qquad r_{0,2}^+ = 5h_{0,3}^2 - 2h_{0,4}, \qquad r_{1,1}^+ = \sqrt{2}(4h_{0,3}h_{1,2} - h_{1,3}), \\
& r_{0,3}^+ = \sqrt{2}\big(-16h_{0,3}^3 + 12h_{0,4}h_{0,3} - 2h_{0,5}\big), \qquad r_{1,0}^+ = -h_{1,2}, \\
& r_{2,0}^+ = \frac{3}{2}h_{1,2}^2 - h_{2,2}, \qquad r_{3,0}^+ = -\frac{5}{2}h_{1,2}^3 + 3h_{2,2}h_{1,2} - h_{3,2}, \\
& r_{1,2}^+ = -35h_{1,2}h_{0,3}^2 + 10h_{1,3}h_{0,3} + 10h_{1,2}h_{0,4} - 2h_{1,4}, \\
& r_{2,1}^+ = \sqrt{2}\big(-12h_{0,3}h_{1,2}^2 + 4h_{1,3}h_{1,2} + 4h_{2,2}h_{0,3} - h_{2,3}\big), \\
& r_{4,0}^+ = -\frac{15}{2}h_{2,2}h_{1,2}^2 + \frac{35}{8}h_{1,2}^4 + 3h_{3,2}h_{1,2} + \frac{3}{2}h_{2,2}^2 - h_{4,2}, \\
& r_{5,0}^+ = \frac{35}{2}h_{2,2}h_{1,2}^3 - \frac{15}{2}h_{3,2}h_{1,2}^2 - \frac{63}{8}h_{1,2}^5 + 3h_{4,2}h_{1,2} - \frac{15}{2}h_{1,2}h_{2,2}^2 + 3h_{3,2}h_{2,2} - h_{5,2}.
\end{aligned}
$$

Let $G(x,w) = d_{0,0} + \sum_{i+j\geqslant 1} n_{i,j}^+ x^i w^j$, $P_3 = G(x,w)y^+/(\sqrt{2}\,w) - S(x, y^+)$. Then from $P_3 = 0$ we have

$$
\begin{aligned}
n_{0,1}^+ &= -2\sqrt{2}h_{0,3}(a_{1,0} + b_{0,1}) + \sqrt{2}(a_{1,1} + 2b_{0,2}), \\
n_{1,0}^+ &= -(a_{1,0} + b_{0,1})h_{1,2} + 2a_{2,0} + b_{1,1}, \\
n_{0,2}^+ &= \big(15h_{0,3}^2 - 6h_{0,4}\big)(a_{1,0} + b_{0,1}) - 6h_{0,3}(a_{1,1} + 2b_{0,2}) + 2a_{1,2} + 6b_{0,3}, \\
n_{2,0}^+ &= \left(\frac{3}{2}h_{1,2}^2 - h_{2,2}\right)(a_{1,0} + b_{0,1}) - h_{1,2}(2a_{2,0} + b_{1,1}) + 3a_{3,0} + b_{2,1}, \\
n_{1,1}^+ &= -2\sqrt{2}(h_{1,3} - 4h_{1,2}h_{0,3})(a_{1,0} + b_{0,1}) - 2\sqrt{2}h_{1,2}(a_{1,1} + 2b_{0,2}) \\
&\quad + 2\sqrt{2}(a_{2,1} + b_{1,2}) - 2\sqrt{2}h_{0,3}(b_{1,1} + 2a_{2,0}), \\
n_{3,0}^+ &= \left(\frac{3}{2}h_{1,2}^2 - h_{2,2}\right)(2a_{2,0} + b_{1,1}) + \left(3h_{2,2}h_{1,2} - \frac{5}{2}h_{1,2}^3 - h_{3,2}\right)(a_{1,0} + b_{0,1}) \\
&\quad - h_{1,2}(3a_{3,0} + b_{2,1}) + 4a_{4,0} + b_{3,1},
\end{aligned}
$$

$$n_{4,0}^{+}=\left(\frac{3}{2}h_{1,2}^{2}-h_{2,2}\right)(3a_{3,0}+b_{2,1})+\left(3h_{2,2}h_{1,2}-\frac{5}{2}h_{1,2}^{3}-h_{3,2}\right)(2a_{2,0}+b_{1,1})$$
$$+\left(-\frac{15}{2}h_{2,2}h_{1,2}^{2}+\frac{35}{8}h_{1,2}^{4}+3h_{3,2}h_{1,2}+\frac{3}{2}h_{2,2}^{2}-h_{4,2}\right)(a_{1,0}+b_{0,1})$$
$$-h_{1,2}(4a_{4,0}+b_{3,1})+5a_{5,0}+b_{4,1},$$
$$n_{5,0}^{+}=\left(\frac{3}{2}h_{1,2}^{2}-h_{2,2}\right)(4a_{4,0}+b_{3,1})+\left(3h_{2,2}h_{1,2}-\frac{5}{2}h_{1,2}^{3}-h_{3,2}\right)(3a_{3,0}+b_{2,1})$$
$$+\left(-\frac{15}{2}h_{2,2}h_{1,2}^{2}+\frac{35}{8}h_{1,2}^{4}+3h_{1,2}h_{3,2}+\frac{3}{2}h_{2,2}^{2}-h_{4,2}\right)(2a_{2,0}+b_{1,1})$$
$$+b_{5,1}+6a_{6,0}+\left(\frac{35}{2}h_{2,2}h_{1,2}^{3}-\frac{15}{2}h_{3,2}h_{1,2}^{2}-\frac{63}{8}h_{1,2}^{5}+3h_{4,2}h_{1,2}\right.$$
$$\left.-\frac{15}{2}h_{1,2}h_{2,2}^{2}+3h_{3,2}h_{2,2}-h_{5,2}\right)(a_{1,0}+b_{0,1})-h_{1,2}(5a_{5,0}+b_{4,1}).$$

Let $x=\sum_{i\geqslant 1}p_iu^i$ and $P_4=H_0+\frac{u^4}{4}$. Then from $P_4=0$ we have

$$p_1=1,\qquad p_2=h_{5,0},\qquad p_3=\frac{7}{2}h_{5,0}^{2}+h_{6,0},\qquad p_4=8h_{6,0}h_{5,0}+16h_{5,0}^{3}+h_{7,0},$$
$$p_5=\frac{117}{2}h_{6,0}h_{5,0}^{2}+\frac{9}{2}h_{6,0}^{2}+h_{8,0}+\frac{663}{8}h_{5,0}^{4}+9h_{7,0}h_{5,0},$$
$$p_6=70\big(h_{5,0}h_{6,0}^{2}+h_{7,0}h_{5,0}^{2}\big)+h_{9,0}+10(h_{7,0}h_{6,0}+h_{8,0}h_{5,0})$$
$$+462h_{5,0}^{5}+420h_{6,0}h_{5,0}^{3}.$$

Hence from (4), (20), by using Maple 10.0 we can obtain the expression of $d_{i,j}$ appeared in Theorem 2. □

3. The existence and number of limit cycles

As stated in Section 1, one can study the number of isolated zeros of the first-order Melnikov function $M(h)$, which is crucial to study the weak Hilbert 16th problem. Moreover, the study of the above problem can give an upper bound of the number of limit cycles of system $(F)_\varepsilon$, which is one of the main topics in the bifurcation theory. Therefore, Melnikov function $M(h)$ plays an important role in studying the existence and number of limit cycles of system $(F)_\varepsilon$ for ε small. See [2,11,13,15]. In this section, we are going to use the expansion of the function $M(h)$ to estimate the number of limit cycles of system $(E)_\varepsilon$. To be precise, we will give the relationship between the first coefficients of the expansion of $M(h)$ and the existence and number of limit cycles of system $(E)_\varepsilon$ in a neighborhood of the nilpotent saddle loop of system $(E)_{\varepsilon=0}$.

Note that the first $3m+1$ coefficients of $M(h)$ in Theorem 1 satisfy

$$c_1=c_{0,m}^{(0)},\qquad c_{k+1}=c_{0,k,m}^{(1)},\quad k=1,\dots,m-1,$$
$$c_{m+1}=c_{1,m}^{(2)},\qquad c_{m+2}=c_{1,m}^{(0)},$$

$$c_{k+2}=c^{(1)}_{0,k,m},\quad k=m+1,\dots,2m-1,$$
$$c_{2m+1+k}=c^{(1)}_{1,k,m},\quad k=1,\dots,m-1,\qquad c_{3m+1}=c^{(2)}_{2,m}.$$

By Theorem 1, it is easy to prove the following theorem by using the method as was shown in [18].

Theorem 4. *Suppose there exists a parameter vector $a_0=(a_{1,0},a_{2,0},\dots,a_{n,0})\in\mathbb{R}^n$ with $n\geqslant 2$ such that*

$$c_i(a_0)=0,\quad i=1,\dots,j\ (j\leqslant 3m),\qquad c_{j+1}(a_0)\neq 0,\qquad \operatorname{rank}\frac{\partial(c_1,\dots,c_j)}{\partial a}\bigg|_{a_0}=j.$$

Then for some (ε,a) near $(0,a_0)$, system $(E)_\varepsilon$ can have j limit cycles near the nilpotent saddle loop of system $(E)_{\varepsilon=0}$.

Obviously Theorem 4 is important for us to study the bifurcation of limit cycle for system $(E)_\varepsilon$. As an application of Theorem 4, we consider the following system.

Example 11.

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=x(x-1)^3+\varepsilon y\big(\mu_1+\mu_2x+\mu_3x^2+x^3\big),\end{aligned}\tag{$(G)_\varepsilon$}$$

where ε is a small parameter and $a=(\mu_1,\mu_2,\mu_3)$ is a parameter vector.

Now taking the transformation $X=1-x$, $Y=y$ and $T=-t$ and still denote X, Y and T by x, y and t, respectively, then system $(G)_\varepsilon$ can be taken into the following form

$$\begin{aligned}\dot{x}&=y=P(x,y),\\ \dot{y}&=x^3(1-x)+\varepsilon g(x,y,a)=Q(x,y),\end{aligned}\tag{$(G_1)_\varepsilon$}$$

where

$$g(x,y,a)=y\big[-(\mu_1+\mu_2+\mu_3+1)+(3+\mu_2+2\mu_3)x-(3+\mu_3)x^2+x^3\big].$$

The Hamiltonian of the unperturbed system $(G_1)_{\varepsilon=0}$ has the form

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{4}x^4+\frac{1}{5}x^5.$$

It has two singular points: nilpotent saddle point $O=(0,0)$ and center $C=(1,0)$. Moreover equation $H(x,y)=0$ corresponds to a nilpotent saddle loop $L=L_1\cup O\cup L_2$, where

$$L_1\colon\ y=\frac{x^2}{10}\sqrt{50-40x},\quad 0\leqslant x\leqslant x_1=\frac{5}{4},$$
$$L_2\colon\ y=-\frac{x^2}{10}\sqrt{50-40x},\quad 0\leqslant x\leqslant x_1=\frac{5}{4}.$$

From Theorems 1–2, we have

$$\begin{aligned} c_1 &= \oint_L g(x,y)\,dx \\ &= -(\mu_1+\mu_2+\mu_3+1)G_{01}+(3+\mu_2+2\mu_3)G_{11}-(3+\mu_3)G_{21}+G_{31}, \end{aligned}$$

where $G_{ij}=\oint_L x^i y^j\,dx$. Using Maple 10.0 we have

$$G_{01}=\frac{25}{84}\sqrt{2},\qquad G_{11}=\frac{125}{504}\sqrt{2},\qquad G_{21}=\frac{625}{2772}\sqrt{2},\qquad G_{31}=\frac{15625}{72072}\sqrt{2}.$$

For c_2, we have

$$c_2=-\frac{4\sqrt{2}\Delta_{0,2}}{3}g_y|_{(0,0)}=\frac{4\sqrt{2}\Delta_{0,2}}{3}(\mu_1+\mu_2+\mu_3+1).$$

For c_3, from Theorems 1–3, we have

$$c_3=\frac{\sqrt{2}d_{1,0}}{2}=\frac{\sqrt{2}}{2}\left(\frac{13}{5}-\frac{2\mu_1}{5}+\frac{3\mu_2}{5}+\frac{8\mu_3}{5}\right).$$

When $c_2=c_3=0$, we have

$$c_4=\oint_L g_y(x,y)\,dt=\oint_L \frac{-(1+\mu_1)x^2+x^3}{y}\,dx=\frac{25\sqrt{2}}{6}-5\sqrt{2}(1+\mu_1).$$

Then the equations $c_1=c_2=c_3=0$ have a unique solution

$$a_0=(\mu_1^*,\mu_2^*,\mu_3^*)=\left(\frac{1}{26},\frac{14}{13},-\frac{53}{26}\right). \tag{38}$$

Substituting (38) into c_4 we have $c_4(a_0)=\frac{25\sqrt{2}}{39}\neq 0$.

By Theorem 4, summarizing the results obtained above gives the following

Theorem 5. *For $|\varepsilon|\ll 1$, there exists some parameter vector $a=(\mu_1,\mu_2,\mu_3)$ near $a_0=(\mu_1^*,\mu_2^*,\mu_3^*)$ such that system $(G)_\varepsilon$ can have at least 3 limit cycles near the nilpotent saddle loop of system $(G)_{\varepsilon=0}$.*

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